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To link to this article: http://dx.doi.org/10.1080/10236198.2016.1216550
On discrete models of fractal dimension to explore the complexity of discrete dynamical systems

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ABSTRACT
A fractal structure is a countable family of coverings which displays accurate information about the irregularities that a set presents when being explored with enough level of detail. It is worth noting that fractal structures become especially appropriate to provide new definitions of fractal dimension, which constitutes a valuable measure to test for chaos in dynamical systems. In this paper, we explore several approaches to calculate the fractal dimension of a subset with respect to a fractal structure. These models generalize the classical box dimension in the context of Euclidean subspaces from a discrete viewpoint. To illustrate the flexibility of the new models, we calculate the fractal dimension of a family of self-affine sets associated with certain discrete dynamical systems.

1. Introduction
One of the topics in discrete dynamical systems theory, which deals with the evolution in discrete time steps of certain quantities over time (several examples are provided in [15,16,19]), consists of the study of certain dynamic invariants associated with expressions of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}),$$

where $f$ is a function involving $k$ continuous variables. The level of complexity concerning such invariants has been explored through several tools to quantify their chaotic behavior. They include the classical box dimension as one of the most applied procedures, though a large collection of definitions and algorithms can be found in literature for that purpose [4,12,17,18].

However, it is worth mentioning that the box dimension, denoted as $\text{dim}_B$ hereafter, can be calculated only on Euclidean subsets. In this paper, we provide several approaches to generalize the box dimension in the context of fractal structures which would allow to calculate the fractal dimension in a wider range of spaces and situations.

A fractal structure is a kind of uniformity that consists of a countable family of coverings that provides more accurate information about the irregularities that a given set presents as it is explored with enough level of detail. This is carried out through a discrete sequence
of stages, called levels. Overall, it holds that fractal structures provide a wonderful context where new definitions of fractal dimension could be provided [7].

The main goal in this work is to properly answer the next question naturally arising. Let $\Gamma$ be a fractal structure on a (metric) space $X$. How could be appropriately defined a fractal dimension function with respect to that fractal structure, say $\dim_{\Gamma}$, such that $\dim_{\Gamma}(F) = \dim_{B}(F)$, provided that $\Gamma$ is fixed as the natural fractal structure (see Definition 3.2) on $F \subseteq \mathbb{R}^d$? Under these assumptions, it becomes obvious that such a fractal dimension function would generalize the box dimension on Euclidean subspaces.

The structure of this paper is as follows. Section 2 contains some mathematical background including quasi-pseudometrics and fractal structures. In Section 3, we explore a $2^{-n}$-cube generalization of the classical box dimension which do not depend on any metric. To deal with, the concept of the natural fractal structure which any Euclidean subset can be always endowed with becomes relevant. Additionally, in Section 4, we develop an extended version of the so-called fractal dimension I which allows the possibility that different diameter sets may appear in each level of the involved fractal structure. Both fractal dimensions I and II do generalize the box dimension in the Euclidean case and it is worth noting that their definitions can be calculated as easy as the box dimension. On the other hand, both Sections 5 and 6 provide some Hausdorff dimension type models for a fractal structure. Thus, fractal dimension III generalizes the box dimension whereas the remaining models do generalize the Hausdorff dimension. The main theoretical properties regarding each fractal dimension approach are summarized at the end of each section. Finally, in Section 7, we calculate the box dimension of a self-affine set which becomes the attractor of a family of discrete dynamical systems. Further, we explain the meaning of the obtained results.

2. Preliminaries

In this section, we recall some definitions, results and notations that will be useful in upcoming sections. We shall focus on quasi-pseudometrics and fractal structures.

2.1. Quasi-pseudometrics

A quasi-pseudometric on a set $X$ is a non-negative real-valued function $\rho$ defined on $X \times X$ such that for all $x, y, z \in X$, the two following are satisfied:

1. $\rho(x, x) = 0$.
2. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Further, if $\rho$ also satisfies that $\rho(x, y) = \rho(y, x) = 0$ if and only if $x = y$, then $\rho$ is said to be a quasi-metric. If $(X, \rho)$ is a (quasi-)metric space, then the diameter of any subset $A \subseteq X$ is defined by $\text{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$, as usual.

2.2. Fractal structures

Fractal structures provide a context where new models for a fractal dimension definition could be defined. In addition, they allow to calculate the fractal dimension in new spaces and situations.
By a covering of a set $X$, we shall understand a family $\Gamma$ of subsets of $X$ such that $X = \bigcup \{A : A \in \Gamma\}$. Let $\Gamma_1$ and $\Gamma_2$ be two coverings of $X$. By $\Gamma_1 \prec \Gamma_2$, we shall denote that $\Gamma_1$ is a refinement of $\Gamma_2$, namely, for all $A \in \Gamma_1$ there exists $B \in \Gamma_2$ such that $A \subseteq B$. Moreover, the notation $\Gamma_1 \ll \Gamma_2$ means that $\Gamma_1 \prec \Gamma_2$ and for all $B \in \Gamma_2$, it holds that $B = \bigcup \{A \in \Gamma_1 : A \subseteq B\}$. Thus, a fractal structure on a set $X$ is defined as a countable family of coverings of $X$, $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$, such that $\Gamma_{n+1} \ll \Gamma_n$ for all $n \in \mathbb{N}$. The covering $\Gamma_n$ is called level $n$ of $\Gamma$.

**Remark 1:** To simplify the theory, the levels of any fractal structure $\Gamma$ will not be coverings in the usual sense. Instead of this, we shall allow that a set can appear twice or more in any level of $\Gamma$. For instance, $\Gamma_1 = \{[0, 1/2], [1/2, 1], [0, 1/2]\}$ may be the first level of a fractal structure defined on the closed unit interval $[0, 1]$.

If $\Gamma$ is a fractal structure on $X$ and $\text{St}(x, \Gamma) = \bigcup \{A \in \Gamma : x \in A\}$ is a neighborhood base for all $x \in X$, then $\Gamma$ is called as a starbase fractal structure. A fractal structure $\Gamma$ is said to be finite if all its levels $\Gamma_n$ are finite coverings. A fractal structure $\Gamma$ is said to be locally finite if for each level $\Gamma_n$ of $\Gamma$, it holds that any point $x \in X$ belongs to a finite number of elements $A \in \Gamma_n$.

It is noteworthy that every metric space admits a compatible fractal structure (see [1, Theorem 4.1] and [2, Theorem 3.18]).

### 3. First approach: counting the number of $2^{-n}$ cubes

First of all, let us recall the standard definition of the box dimension.

**Definition 3.1:** ([6], Equivalent definitions 2.1) The (lower/upper) box dimension of a subset $F$ of $\mathbb{R}^d$ is given as the following (lower/upper) limit:

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

where $N_\delta(F)$ can be calculated as one of the quantities listed below:

1. The number of $\delta$-cubes that intersect $F$.
2. The smallest number of sets of diameter at most $\delta$ that cover $F$.
3. The largest number of disjoint balls of radius $\delta$ with centres in $F$.
4. The smallest number of closed balls of radius $\delta$ that cover $F$.
5. The smallest number of $\delta$-cubes that cover $F$.

A proof regarding the equivalence of all the descriptions provided above for $N_\delta(F)$ could be found out in Subsection 2.1 of Falconer’s book [6].

The following remark becomes useful for upcoming calculations.

**Remark 2:** A suitable discretization regarding the continuum of scales $\delta$ could be carried out for box dimension calculation purposes. More specifically, it is possible to calculate the (lower/upper) box dimension of a subset $F$ of $\mathbb{R}^d$ via any decreasing sequence $\{\delta_n : n \in \mathbb{N}\}$ such that $c \cdot \delta_n \leq \delta_{n+1}$, where $c \in (0, 1)$ is a constant. In particular, it can be deal with by $\delta_n = c^n$.

Next, we define a locally finite starbase fractal structure that each Euclidean subspace can be always endowed with.
Definition 3.2: ([9], Definition 3.1) The natural fractal structure on the Euclidean space $\mathbb{R}^d$ is the countable family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$, whose levels are

$$
\Gamma_n = \left\{ \left[ \frac{k_1}{2^n}, \frac{k_1 + 1}{2^n} \right] \times \cdots \times \left[ \frac{k_d}{2^n}, \frac{k_d + 1}{2^n} \right] : k_1, \ldots, k_d \in \mathbb{Z} \right\}.
$$

It is worth noting that the natural fractal structure on $\mathbb{R}^d$ is a tiling consisting of $2^{-n}$-cubes on $\mathbb{R}^d$. Thus, let us explain how to define a discrete box dimension type model whose range of applications goes beyond Euclidean subsets.

Fractal dimension approach 3.3: Let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on $X = \mathbb{R}^d$, and assume that the order of the diameters in each level $n$ of $\Gamma$ is equal to $2^{-n}$. Further, let $c = 1/2$. Hence, by Remark 2, we can choose $\delta_n = 2^{-n}$ as a suitable decreasing sequence. According to Definition 3.1 (1), it holds that $N_{2^{-n}}(F)$ is the number of $2^{-n}$-cubes that intersect $F \subseteq \mathbb{R}^d$. Moreover, let us define $N_n(F)$ as the number of elements in level $n$ of $\Gamma$ that intersect $F$, namely, $N_n(F) = \text{Card}(\mathcal{A}_n(F))$, where $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$, and Card $(A)$ being the number of elements in $A$. Thus, $N_{2^{-n}}(F) = N_n(F)$. Next step is to explore the logarithmic rate at which $N_n(F)$ increases as $n \rightarrow \infty$. This leads to a discrete version of the underlying idea in the box dimension case, where it is analyzed how $N_n(F)$ increases as $\delta \rightarrow 0$ (also in a logarithmic scale). Hence, a gradient of the graph involving $\log 2^{-n}$ vs. $\log N_n(F)$ gives a register regarding fractal patterns in $F$.

That idea can be empirically extended to non-Euclidean contexts where the standard box dimension can have no sense or cannot be calculated [13]. Thus, both Definition 3.1 (1) and Remark 2 lead to the next key definition.

Definition 3.4: ([9], Definition 3.3) Let $\Gamma$ be a fractal structure on $X$ and $F$ be a subset of $X$. Moreover, let $N_n(F)$ be the number of elements in level $n$ of $\Gamma$ that intersect $F$. The (lower/upper) fractal dimension $I$ of $F$ is given as the following (lower/upper) limit:

$$
\dim^I_\Gamma(F) = \lim_{n \to \infty} \frac{1}{n} \log_2 N_n(F).
$$

Next, we explain how to calculate the diameter of any level of a fractal structure as well as the diameter of a subset in a level of a fractal structure. To deal with, let $(X, \rho)$ be a (quasi-)metric space. Thus, the diameter of a subset $A$ of $X$ is given by $\text{diam} (A) = \sup \{\rho (x, y) : x, y \in A\}$, as usual.

Definition 3.5: ([9], Definition 3.7) Let $\Gamma$ be a fractal structure on a distance space $(X, \rho)$, and $F$ be a subset of $X$.

1. The diameter of level $n$ of $\Gamma$ is given by $\delta(\Gamma_n) = \sup \{\text{diam} (A) : A \in \Gamma_n\}$.
2. The diameter of $F$ in level $n$ of $\Gamma$ is defined as $\delta(F, \Gamma_n) = \sup \{\text{diam} (A) : A \in \mathcal{A}_n(F)\}$.

The following result collects some properties regarding the fractal dimension $I$ behavior as a fractal dimension function. The proofs therein can be found out in Section 3 of [9].

Theorem 3.6: Let $(X, \Gamma)$ be a GF-space.

1. If $E \subseteq F$, then $\dim^I_\Gamma(E) \leq \dim^I_\Gamma(F)$. This inequality stands for both lower and upper fractal dimensions $I$. 


Let \( \{E_i : i = 1, \ldots, n\} \) be a finite family of subsets of \( X \). Then the upper fractal dimension \( I \) is finitely stable, namely, \( \dim_1(\bigcup_{i=1}^n E_i) = \max\{\dim_1(E_i) : i = 1, \ldots, n\} \).

Neither lower nor upper fractal dimension \( I \) are countably stable, namely, it is not satisfied, in general, that \( \dim_1(\bigcup_{i \in I} E_i) = \max\{\dim_1(E_i) : i \in I\} \), where \( \{E_i : i \in I\} \) is a countable sequence of subsets of \( X \).

There exist a countable subset \( F \) of \( X \) and a fractal structure \( \Gamma \) on \( X \) such that \( \dim_1(F) \neq 0 \).

There exist a subset \( F \) of \( X \) and a locally finite starbase fractal structure \( \Gamma \) on \( X \) such that \( \dim_1(F) \neq \dim_1(\overline{F}) \).

If \( \Gamma \) is the natural fractal structure on \( X = \mathbb{R}^d \) and \( F \) is any subset of \( X \), then \( \dim_1(B(F)) = \dim_1(F) \).

Let us assume that \( X \) is endowed with a metric \( \rho \) and \( F \) is any subset of \( X \). If there exists \( c \in (0, 1) \) such that \( \delta(F, \Gamma_{n+1}) \leq c \cdot \delta(F, \Gamma_n) \), then the following inequality holds for both the lower/upper fractal dimensions \( I \) of \( F \): \( \dim_1(B(F)) \leq \kappa_c \cdot \dim_1(F) \), where \( \kappa_c \) is a constant which depends on \( c \).

Let \( \mathcal{F} = \{f_1, \ldots, f_k\} \) be an IFS (see upcoming Section 7) on a complete metric space \( X \), \( \mathcal{K} \) be its IFS-attractor, \( c_i \) be the similarity ratio associated with each similarity \( f_i \), and assume, in addition, that \( \Gamma \) is the natural fractal structure on \( X \) as a self-similar set (see [3, Definition 4.4]). Then \( \dim_1(B(K)) \leq \kappa_c \cdot \dim_1(F(K)) \), where \( \kappa_c \) is a constant which depends on \( c = \max(c_i : i = 1, \ldots, k) \).

Let \( \Gamma_1 \neq \Gamma_2 \) be fractal structures on \( X \). There exists a subset \( F \) of \( X \) such that \( \dim_1(\Gamma_1(F)) \neq \dim_1(\Gamma_2(F)) \).

It is worth noting that fractal dimension \( I \) treats all the elements in a same level of a fractal structure as having the same size (or at least, as being of the same order), namely, \( 2^{-n} \). Moreover, fractal dimension \( I \) generalizes the classical box dimension on Euclidean subspaces (see Theorem 3.6 (6)). Further, it holds that fractal dimension \( I \) depends on a fixed fractal structure for calculation purposes, as Theorem 3.6 (9) points out.

A fractal structure plays the role of a uniform structure. In fact, if there is no metric available, then we can ‘measure’ subsets depending on what level of the fractal structure they belong to. In this sense, it results quite natural that fractal dimension \( I \) depends on a fractal structure, whereas the box dimension depends on a metric.

4. Second model: involving a distance function

So far, the fractal dimension model provided in Definition 3.4 only takes into account a fractal structure on \( X \) for fractal dimension calculation purposes. Moreover, that definition someway considers that all the elements in level \( n \) of the fixed fractal structure have a diameter (whose order is) equal to \( 2^{-n} \). Thus, the natural fractal structure on any Euclidean space \( \mathbb{R}^d \) (as given in Definition 3.2), that is actually applied for box dimension calculation purposes, could be further extended to other kinds of fractal structures, such as tilings (e.g. triangulations) on \( X \). In this way, recall that any compact surface can be endowed with a triangulation.

Next step is to allow that our generalized definition of fractal dimension with respect to a fractal structure enables the possibility that different diameter elements could appear in any level of a fractal structure. Thus, one option is to weigh the number of elements in
level $n$ of a fractal structure that intersect $F$ by the largest of all of them, in terms of the quantity $\delta(F, \Gamma_n)$.

Accordingly, the main purpose herein is to consider a fractal structure on $X$ simultaneously with some kind of metric in that space. That metric will allow to ‘measure’ the size of the elements in any level of the fractal structure. For instance, for any Euclidean IFS-attractor, we can use both the Euclidean metric as well as the natural fractal structure on the attractor as a self-similar set itself.

To deal with, let us consider the most general concept of a metric function, namely, a distance function. In fact, let $\rho : X \times X \longrightarrow \mathbb{R}$ be a non-negative real valued function. It is said to be a distance function provided that $\rho(x, x) = 0$ for all $x \in X$. We shall calculate diameters of subsets, coverings, ..., etc, similarly as in the case of a metric. Given this, next we provide the second key definition of fractal dimension for a fractal structure.

**Definition 4.1: [9], Definition 4.2** Let $\Gamma$ be a fractal structure on a distance space $(X, \rho)$, and $F$ be a subset of $X$. Moreover, let $N_n(F)$ be the number of elements in level $n$ of $\Gamma$ that intersect $F$, and $\delta(F, \Gamma_n)$ be as in Definition 3.5 (2), as well. The (lower/upper) fractal dimension $\dim_\Gamma$ of $F$ is given as the following (lower/upper) limit:

$$\dim_\Gamma^\downarrow(F) = \lim_{n \to \infty} \frac{\log N_n(F)}{-\log \delta(F, \Gamma_n)}.$$  

The following result contains some properties regarding fractal dimension II. We refer reader to Section 4 of [9] for their proofs.

**Theorem 4.2:** Let $\Gamma$ be a fractal structure on a distance space $(X, \rho)$.

1. If $E \subseteq F$, then $\dim_\Gamma^\downarrow(E) \leq \dim_\Gamma^\downarrow(F)$. This inequality holds for both lower and upper fractal dimensions II.

2. There exist two subsets $E_1$ and $E_2$ of $X$ and a fractal structure $\Gamma$ such that $\dim_\Gamma^\downarrow(E_1 \cup E_2) \neq \max\{\dim_\Gamma^\downarrow(E_1), \dim_\Gamma^\downarrow(E_2)\}$, namely, fractal dimension II is not finitely stable.

3. Neither lower nor upper fractal dimension II are countably stable, namely, it is not satisfied, in general, that $\dim_\Gamma^\downarrow\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup\{\dim_\Gamma^\downarrow(E_i) : i = 1, \ldots, \infty\}$, where $\{E_i : i = 1, \ldots, \infty\}$ is a countable sequence of subsets of $X$.

4. There exist a countable subset $F$ of $X$ and a fractal structure $\Gamma$ on $X$ such that $\dim_\Gamma^\downarrow(F) \neq 0$.

5. There exist a subset of $F$ of $X$ and a locally finite starbase fractal structure $\Gamma$ on $X$ such that $\dim_\Gamma^\downarrow(F) \neq \dim_\Gamma^\uparrow(F)$.

6. Let $\rho$ be the semimetric associated with the fractal structure $\Gamma$ (see [3, Theorem 6.5]) and $F$ be a subset of $X$. If $\Gamma$ is starbase and for all level $n$ there exists $x \in F$ such that $\text{St}(x, \Gamma_n) \neq \text{St}(x, \Gamma_{n+1})$, then $\dim_\Gamma^\downarrow(F) = \dim_\Gamma^\downarrow(F)$, provided that these fractal dimensions exist. Otherwise, such an equality yields for both lower/upper fractal dimensions I and II.

7. If $\Gamma$ is the natural fractal structure on $X = \mathbb{R}^d$ and $F$ is any subset of $X$, then $\dim_\Gamma^\downarrow(F) = \dim_\Gamma^\downarrow^\downarrow(F) = \dim_\Gamma^\downarrow(F)$.

8. If $F$ is a subset of $X$ and diam $(F, \Gamma_n) \rightarrow 0$, then

   (a) $\dim_H(F) \leq \dim_{\Gamma BFS}(F) \leq \dim_{\Gamma BS}(F)$.

   (b) If there exist both $\dim_{\Gamma BFS}(F)$ and $\dim_{\Gamma BS}(F)$, then $\dim_{\Gamma BFS}(F) \leq \dim_{\Gamma BS}(F)$.
(c) If there exists a constant $c > 0$ such that $\delta(F, \Gamma_n) \leq c \cdot \delta(F, \Gamma_{n+1})$. Then 
\[ \dim_{B}(F) \leq \dim_{F}^{2}(F). \]

(d) If $\Gamma$ is under the $\kappa$-condition (see [9, Theorem 4.13]), then

(i) $\dim_{B}(F) \leq \dim_{F}^{2}(F) \leq \dim_{\Gamma}(F) \leq \dim_{B}(F)$. In addition, if $\dim_{B}(F)$ exists, then $\dim_{B}(F) = \dim_{F}^{2}(F)$.

(ii) If there exists $c \in (0, 1)$ such that $c \cdot \delta(F, \Gamma_n) \leq \delta(F, \Gamma_{n+1})$, then 
\[ \dim_{B}(F) = \dim_{F}^{2}(F) \text{ and } \dim_{B}(F) = \dim_{\Gamma}(F), \text{ as well.} \]

(9) Let $F = \{f_1, \ldots, f_k\}$ be a family of contractions on $\mathbb{R}^d$, $K$ be its IFS-attractor, $c_i$ be the similarity ratio associated with each similarity $f_i$, and assume, in addition, that $\Gamma$ is the natural fractal structure on $K$ as a self-similar set. For all $F \subseteq K$, the four following hold:

(a) $\dim_{B}(F) \leq \dim_{\Gamma}^{2}(F)$.

(b) If there exist both $\dim_{B}(F)$ and $\dim_{\Gamma}^{2}(F)$, then $\dim_{B}(F) \leq \dim_{\Gamma}^{2}(F)$.

(c) If some $f_i \in F$ is a bilipschitz contraction, then $\dim_{B}(F) \leq \dim_{\Gamma}^{2}(F)$. In particular, this inequality yields if $K$ is self-similar.

(d) If all the $f_i$ are similitudes, $F$ is under the OSC (see [6, Section 9.2]), and all the similarity ratios are equal to $r$, then $\dim_{B}(K) = \dim_{\Gamma}^{2}(K) = - \log k / \log r$.

(10) Assume that $\diam(A) = \delta(F, \Gamma_n)$ for all $A \in A_n(F)$, where $O(\delta(F, \Gamma_n)) = O(2^{-n})$. 
Then $\dim_{\Gamma}^{1}(F) = \dim_{\Gamma}^{2}(F)$.

It is worth noting that fractal dimension I only depends on a fractal structure, whereas fractal dimension II also depends on a metric. To illustrate the difference between these models of fractal dimension for a fractal structure, let $C$ be the standard middle third Cantor set and denote by $C_i$ slight modifications of $C$ we state next. Let us assume that the contraction ratio of each subset $C_i$ is $c_i \in [1/3, 1/2)$ for the two similitudes that generate $C_i$. If $\Gamma_i$ is the natural fractal structure on $C_i$ as a self-similar set, then $\delta(C_i, \Gamma_n) = c_i^n$, which leads to \[ \dim_{B}(C_i) = \dim_{\Gamma}^{2}(C_i) = \log 2 / - \log c_i \rightarrow 1 = \dim_{\Gamma}^{1}(C), \text{ provided that} \]

\[ c_i \rightarrow 1/2. \]

In addition, by Theorem 4.2 (7), both box dimension and fractal dimension I remain as particular cases of fractal dimension II. Further, observe that such a result allows the calculation of the box dimension of a subset by counting triangles instead of squares, for instance, since a fractal structure with levels consisting of triangles whose diameters are (of the order of) $2^{-n}$ can be always defined on the plane.

## 5. Third approach: a discretized Hausdorff dimension

The classical Hausdorff dimension constitutes the oldest and also the most accurate model of fractal dimension. In this section, we show how that classical model for fractal dimension allows the definition of a discrete model of fractal dimension with respect to a fractal structure.

**Fractal dimension approach 5.1:** Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$ and $F$ be a subset of $X$. We shall consider the actual size of all the elements in any level of $\Gamma$ that intersect $F$ via its diameter. Thus, let us consider the family $A_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$, which gives a $\delta(F, \Gamma_n)$-covering of $F$, since $\diam(A) \leq \delta(F, \Gamma_n)$ for all $A \in A_n(F)$. If $s$ is a non-negative real number, then the sum of the $s$-powers of the diameters of all the elements in $A_n(F)$ could provide information regarding fractal patterns in $F$. More specifically, let
\( \mathcal{H}_n^s(F) = \sum_{i \in I} \{ \text{diam} (A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_n(F) \} \) and define also \( \mathcal{H}_n^s(F) = \lim_{n \to \infty} \mathcal{H}_n^s(F) \). If \( t \) is another non-negative real number, then it holds that \( \sum_{A \in \mathcal{A}_n(F)} \text{diam} (A)^t \leq \delta(F, \Gamma_n)^t \cdot \sum_{A \in \mathcal{A}_n(F)} \text{diam} (A)^s \). Equivalently, \( \mathcal{H}_n^t(F) \leq \delta(F, \Gamma_n)^t \cdot \mathcal{H}_n^s(F) \) for all \( n \in \mathbb{N} \).

Letting \( n \to \infty \), we have \( \mathcal{H}_n^t(F) \leq \mathcal{H}_n^s(F) \cdot \lim_{n \to \infty} \delta(F, \Gamma_n)^t \cdot \mathcal{H}_n^s(F) \). Accordingly, if \( \mathcal{H}_n^t(F) < \infty \) and \( \delta(F, \Gamma_n) \to 0 \) provided that \( t > s \), then \( \mathcal{H}_n^t(F) = 0 \). Hence, the critical point \( s \) where \( \mathcal{H}_n^t(F) \) ‘jumps’ from \( \infty \) to zero plays a similar role to the Hausdorff dimension. In fact, such a value equals the quantities \( \sup \{ s : \mathcal{H}_n^s(F) = \infty \} = \inf \{ s : \mathcal{H}_n^s(F) = 0 \} \).

The condition \( \delta(F, \Gamma_n) \to 0 \), though not being too restrictive, since it requires a geometric decreasing regarding the diameters of all the levels in \( \Gamma \), must be considered in the definition of that discrete model. Indeed, as Remark 4.1 in [8] points out, that condition cannot be removed.

It is also worth noting that the set function \( \mathcal{H}_n^s \) plays a similar role as the Hausdorff measure \( \mathcal{H}_n^s \). However, the Hausdorff dimension still keeps some advantages, including the fact that the \( s \)-dimensional Hausdorff measure of \( F \), namely, \( \lim_{\delta \to 0} \mathcal{H}_n^s(F) \), always exists. Such a value lies in \([0, \infty]\) and usually belongs to \([0, \infty)\). Nevertheless, the sequence \( \{ \mathcal{H}_n^s(F) \}_{n \in \mathbb{N}} \) is not monotonic in \( n \in \mathbb{N} \), which implies that \( \mathcal{H}_n^s(F) \) does not exist, in general.

To fix that issue, one option is to replace the families \( \mathcal{A}_n(F) \) by \( \mathcal{A}_{n,3}(F) = \bigcup \{ \mathcal{A}_k(F) : k \geq n \} \), for which the previous arguments still remain valid. They key definition is stated next.

**Definition 5.2:** (8, Remark 4.3) Let \( \Gamma \) be a fractal structure on a metric space \((X, \rho)\), \( F \) a subset of \( X \), and assume, in addition, that \( \delta(F, \Gamma_n) \to 0 \). Moreover, let us define \( \mathcal{H}_{n,3}^s(F) \) as one of the following equivalent expressions:

1. \( \inf \{ \mathcal{H}_n^k(F) : k \geq n \} \)
2. \( \inf \{ \sum_{A \in \mathcal{A}_n(F)} \text{diam} (A)^s : k \geq n \} \)
3. \( \inf \{ \sum_{A \in \mathcal{C}} \text{diam} (A)^s : \mathcal{C} \in \mathcal{A}_{n,3}(F) \} \)

If \( \mathcal{H}_3^s(F) = \lim_{n \to \infty} \mathcal{H}_{n,3}^s(F) \), then the fractal dimension III of \( F \) is the critical point

\[
\dim_3^F(F) = \sup \{ s : \mathcal{H}_3^s(F) = \infty \} = \inf \{ s : \mathcal{H}_3^s(F) = 0 \}
\]

From Definition 5.2, it holds that

\[
\mathcal{H}_3^s(F) = \begin{cases} 
\infty & \text{if } s < \dim_3^F(F) \\
0 & \text{if } s > \dim_3^F(F)
\end{cases}
\]

namely, the set function \( \mathcal{H}_3^s \) behaves like the \( s \)-dimensional Hausdorff measure. It is also worth noting that the sequence \( \{ \mathcal{H}_{n,3}^s(F) \}_{n \in \mathbb{N}} \) is monotonic in \( n \in \mathbb{N} \), and hence, the fractal dimension III of \( F \) exists for all subset \( F \) of \( X \).

Some properties regarding fractal dimension III are stated below. Their proofs can be found in [8].

**Theorem 5.3:** Let \( \Gamma \) be a fractal structure on a metric space \((X, \rho)\), \( F \) a subset of \( X \), and assume that \( \delta(F, \Gamma_n) \to 0 \).

1. If \( E \subseteq F \), then \( \dim_3^\Gamma(E) \leq \dim_3^\Gamma(F) \).
2. Fractal dimension III is not countably stable, namely, it is not satisfied, in general, that \( \dim_3^\Gamma(\bigcup_{i=1}^\infty E_i) = \sup \{ \dim_3^\Gamma(E_i) : i = 1, \ldots, \infty \} \), where \( \{E_i : i = 1, \ldots, \infty\} \) is a countable sequence of subsets of \( X \).
3. There exist a countable subset \( F \) of \( X \) and a fractal structure \( \Gamma \) on \( X \) such that \( \dim_3^\Gamma(F) \neq 0 \).
4. There exist a subset $F$ of $X$ and a locally finite tiling starbase fractal structure $\Gamma$ with finite order on $X$ such that $\dim_D^1(F) \neq \dim_D^2(F)$.

5. $\dim_H(F) \leq \dim_F^1(F) \leq \dim_F^2(F) \leq \dim_F^3(F)$.

6. If $\Gamma$ is the natural fractal structure on $X = \mathbb{R}^d$, then $\dim_B(F) = \dim_D^1(F) = \dim_D^3(F)$.

7. Additionally, if $\text{diam}(A) = \delta(F, \Gamma_n)$ for all $A \in A_n(F)$, then
   (a) $\dim_H(F) \leq \dim_B^1(F) \leq \dim_F^1(F) \leq \dim_F^2(F) \leq \dim_F^3(F)$.
   (b) If $\mathcal{O}(\delta(F, \Gamma_n)) = \mathcal{O}(2^{-n})$, then $\dim_D^1(F) = \dim_F^2(F) = \dim_D^3(F)$.
   (c) If $\Gamma$ is under the $\kappa$-condition, then $\dim_B(F) = \dim_D^1(F)$.

8. Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be a family of similitudes on a complete metric space $X$, $\mathcal{K}$ be its IFS-attractor, $c_i$ be the similarity ratio associated with each similarity $f_i$, and assume, in addition, that $\Gamma$ is the natural fractal structure on $\mathcal{K}$ as a self-similar set. Then
   (a) $\dim_D^1(\mathcal{K}) = s$, where $s$ is the similarity dimension of $\mathcal{K}$, and it also holds that $\mathcal{H}_s^3(\mathcal{K}) \in (0, \infty)$.
   (b) If $\Gamma$ is under the OSC, then $\dim_H(F) = \dim_D^3(\mathcal{K}) = \dim_B(\mathcal{K})$.

6. Fourth stage: improving the accuracy of fractal dimension III

In previous section, the fractal dimension III of a subset $F$ of a (metric) space $X$ was properly defined with respect to a fractal structure. To deal with, it was explored the asymptotic behavior of $\mathcal{H}_{s,3}^n(F)$, and hence, the critical value where $\mathcal{H}_{s}^3(F)$ 'jumps' from $\infty$ to 0 gives such a fractal dimension, which indeed constitutes a generalization of the box dimension on Euclidean subspaces. Next step is to improve the accuracy of fractal dimension III in the sense of generalizing the Hausdorff dimension model in the context of fractal structures.

Fractal dimension approach 6.1: Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, $F$ be a subset of $X$, and assume that $\delta(F, \Gamma_n) \to 0$. It is worth mentioning that the topology induced by the fractal structure $\Gamma$ is usually the same as the topology induced by the metric $\rho$ on $X$. In this way, we shall refer always to the topology induced by $\Gamma$, since both topologies do not have to be the same, in general.

On the other hand, recall that the calculation of $\mathcal{H}_{s,3}^n(F)$ consists of minimizing the sum of the $s$-powers of the diameters of all the elements in an appropriate $\delta(F, \Gamma_n)$-cover of $F$, say $\{A_i\}_{i \in I}$, where all the elements $A_i$ belong to a same level of $\Gamma$ deeper than level $n$. Mathematically, there exists $m \geq n$ such that $A_i \in \Gamma_m$, for all $i \in I$. A further consideration is to allow that given a level $n$ of $\Gamma$, each element $A_i$ may lie in a level deeper than $n$, though not always being the same, necessarily. In other words, for all $i \in I$, there exists $m(i) \geq n$ such that $A_i \in \Gamma_{m(i)}$. Accordingly, let us define the following collection of $\delta(F, \Gamma_n)$-coverings of $F$:

$$B_n(F) = \left\{ \{A_i\}_{i \in I} : A_i \in \bigcup_{i \geq n} \Gamma_i, F \subseteq \bigcup_{i \in I} A_i \right\}. \quad (1)$$

Hence, let $s$ be a non-negative real number and consider, additionally,

$$D_{n,s}^s(F) = \inf \left\{ \sum_{i \in I} \text{diam}(A_i)^s : \{A_i\}_{i \in I} \in B_n(F) \right\}. \quad (2)$$
If $D^s(F) = \lim_{n \to \infty} D^s_n(F)$, then it holds that the set function $D^s$ behaves similarly to the $s$-dimensional Hausdorff measure. To prove that, let $t \geq 0 : t > s$. Thus, $\sum \text{diam} (A_i)^t \leq \delta(F, \Gamma_n)^{t-s} \cdot \sum \text{diam} (A_i)^s$, where in the previous sums, $A_i \in \{A_i\}_{i \in I} \in B_n(F)$. Hence, $D^t_n(F) \leq \delta(F, \Gamma_n)^{t-s} \cdot D^s_n(F)$. Letting $n \to \infty$, we have $D^t(F) \leq D^s(F) \cdot \lim_{n \to \infty} \delta(F, \Gamma_n)^{t-s}$. Accordingly, if $D^t(F) < \infty$, then $D^t(F) = 0$, since $t > s$ and $\delta(F, \Gamma_n) \to 0$, by hypothesis. This implies that the equality $\sup \{s : D^s(F) = \infty\} = \inf \{s : D^s(F) = 0\}$ throws a critical value, which could be defined as a new fractal dimension for a fractal structure involving a discretization regarding the Hausdorff dimension.

The following remark points out that the condition $\delta(F, \Gamma_n) \to 0$ becomes necessary for fractal dimension calculation purposes.

**Counterexample 6.2:** There exist a fractal structure $\Gamma$ on a metric space $(X, \rho)$ and a subset $F$ of $X$ such that $\delta(F, \Gamma_n) \to 0$ and $\sup \{s : D^s(F) = \infty\} \neq \inf \{s : D^s(F) = 0\}$.

**Proof:** Indeed, let $F = (0, 1] \subset \mathbb{R}$ and $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on $[0, 1]$ whose levels are defined as follows:

$$\Gamma_n = \{[0, 1]\} \cup \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] : k = 1, \ldots, 2^n - 1 \right\}.$$  \hspace{1cm} (3)

Since $\delta(F, \Gamma_n) = 1$ for all $n \in \mathbb{N}$, then we have $\delta(F, \Gamma_n) \to 0$. Additionally, the following identity holds:

$$D^s(F) = \begin{cases} 1 & \text{if } s \leq 1 \\ 0 & \text{if } s > 1. \end{cases}$$

In fact,

(1) if $s \leq 1$, then $D^s(F) = 1$, since one of the two following cases may occur:

(a) Assume that the covering of $F$ we choose for fractal dimension calculation purposes contains the interval $[0, 1]$. Thus,

(i) if that covering is $\{[0, 1]\}$, then $D^s_n(F) = (\text{diam} ([0, 1]))^s \leq 1$, and hence, $D^s(F) \leq 1$.

(ii) Assume that the covering of $F$ we choose, $\{A_i\}_{i \in I}$, contains $[0, 1]$ as well as some elements of the natural fractal structure on $[0, 1]$. Then $\sum_{i \in I} (\text{diam} (A_i))^s \geq 1$. Accordingly, $D^s(F) \leq 1$.

(b) On the other hand, assume that the covering of $F$ we select, $\{A_i\}_{i \in I}$, does not contain $[0, 1]$. Thus, notice that such a covering consists of some elements of the natural fractal structure on $[0, 1]$. Since the fractal dimension of $F$ with respect to that natural fractal structure is equal to 1, then $\sum_{i \in I} (\text{diam} (A_i))^s \geq 1$. Hence, $D^s(F) = 1$.

(2) On the other hand, if $s > 1$, then $D^s(F) = 0$. Indeed, if $6 = \{\Sigma_n : n \in \mathbb{N}\}$ denotes the natural fractal structure induced on $[0, 1]$, then it becomes clear that $D^s(F) = H^s(F) = 0$, due to [10, Theorem 3.10]. Let $\varepsilon > 0$ be a fixed but arbitrarily chosen real number. Thus, there exists a covering $\{A_i\}_{i \in I} \in B_n(F)$, such that for all $i \in I$, it holds that $A_i \in \Sigma_k : k \geq n$, and satisfying that $\sum_{i \in I} \text{diam} (A_i)^s < \varepsilon$. One of the two following may occur:

- $A_i \in \Gamma_k : k \geq n$, or
\( A_i = \left[ 0, \frac{1}{2^k} \right] \not\subseteq \Gamma_k : k \geq n. \) In this case, observe that

\[
\left[ 0, \frac{1}{2^k} \right] = \bigcup_{\alpha \geq 1} \left[ \frac{1}{2^{k+\alpha}}, \frac{1}{2^{k+\alpha-1}} \right].
\]

Accordingly, a new covering \( B \) of \( F \) can be constructed from all the elements in \( \{A_i\}_{i \in I} \) but replacing the elements of the form \( \left[ 0, \frac{1}{2^k} \right] \) by

\[
\left[ \frac{1}{2^{k+\alpha}}, \frac{1}{2^{k+\alpha-1}} \right] : \alpha \geq 1, \text{ instead. Further, for each element of the form } \left[ 0, \frac{1}{2^k} \right],
\]

we have

\[
\sum_{\alpha = 1}^{+\infty} \frac{1}{(2^{k+\alpha})^s} = \frac{1}{2^{ks}} \sum_{\alpha = 1}^{+\infty} \frac{1}{(2^s)^{\alpha}} = \frac{1}{2^{ks}} \cdot \frac{1}{2^s - 1} < \frac{1}{2^{ks}} = \left( \text{diam} \left( \left[ 0, \frac{1}{2^k} \right] \right) \right)^s.
\]

Thus, \( \sum_{B \in B} \text{diam} (B)^s \leq \sum_{i \in I} \text{diam} (A_i)^s < \varepsilon, \) leading to \( D_k^s (F) < \varepsilon. \) Hence, \( D_k^s (F) = 0 \) for all \( s > 1. \)

**Fractal dimension approach 6.3:** Similarly to both Equations (1) and (2), the following expressions lead to a discrete fractal dimension for finite coverings, which becomes especially appropriate in empirical applications [11]. In fact, let us define

\[
L_n (F) = \left\{ \{A_i\}_{i \in I} : A_i \in \bigcup_{i \geq n} \Gamma_i \text{ for all } i \in I, F \subseteq \bigcup_{i \in I} A_i, \text{ Card} (I) < \infty \right\},
\]

as well as

\[
K_n^s (F) = \inf \left\{ \sum_{i \in I} \text{diam} (A_i)^s : \{A_i\}_{i \in I} \in L_n (F) \right\}.
\]

(4)

Then the asymptotic behavior of Equation (4) plays a similar role to Hausdorff measure. Let \( K^s (F) = \lim_{n \to \infty} K_n^s (F). \)

The following remark becomes similar to Counterexample 6.2.

**Counterexample 6.4:** There exist a fractal structure \( \Gamma \) on a metric space \( (X, \rho) \) and a subset \( F \) of \( X \) such that \( \delta (F, \Gamma_n) \to 0 \) and \( \sup \{s : K^s (F) = \infty \} \neq \inf \{s : K^s (F) = 0 \}. \)

**Proof:** Let \( F = (0, 1] \subset \mathbb{R} \) and \( \Gamma \) be a fractal structure whose levels are given as in Equation (3). Thus, any finite covering of \( F \) via elements of \( \Gamma \) must contain the interval \([0, 1].\) This implies that \( K^s (F) = 1 \) for all \( s > 0, \) and hence, \( \{s : K^s (F) = \infty \} = \{s : K^s (F) = 0 \} = \emptyset. \)

**Fractal dimension approach 6.5:** Another fractal dimension model described in terms of fractal structures can be sketched as follows. Let \( (X, \rho) \) be a metric space, \( F \) be a subset of \( X, \) and \( \delta > 0. \) Moreover, let us consider the next family of coverings of \( F: \)

\[
G_\delta (F) = \left\{ \{A_i\}_{i \in I} : A_i \in \bigcup_{i \in \mathbb{N}} \Gamma_i \text{ for all } i \in I, \text{diam} (A_i) \leq \delta, F \subseteq \bigcup_{i \in I} A_i \right\},
\]

as well as the expression that follows:
\[ J^s_\delta(F) = \inf \left\{ \sum_{i \in I} \operatorname{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{G}_\delta(F) \right\}. \]

The asymptotic behavior of \( J^s_\delta(F) \) is studied via the following expression:

\[ J^s(F) = \lim_{\delta \to 0} J^s_\delta(F). \]

Let \( t \geq 0 \). Thus,

\[ \sum_{i \in I} \operatorname{diam}(A_i)^t \leq \delta^{t-s} \cdot \sum_{i \in I} \operatorname{diam}(A_i)^s, \tag{5} \]

where the sums are considered on \( \mathcal{G}_\delta(F) \). Taking infima in Equation (5), we have

\[ J^t_\delta(F) \leq \delta^{t-s} \cdot J^s_\delta(F). \]

Hence,

\[ J^t(F) \leq J^s(F) \cdot \lim_{\delta \to 0} \delta^{t-s}. \]

Therefore, if \( J^s(F) < \infty \) and \( \delta \to 0 \) provided that \( t > s \), then it holds that \( J^s(F) = 0 \). Accordingly, the critical point where \( J^s(F) \) 'jumps' from \( \infty \) to zero throws a fractal dimension of \( F \), namely,

\[ \sup\{s : J^s(F) = \infty\} = \inf\{s : J^s(F) = 0\}. \]

The previous arguments are formalized next.

**Definition 6.6:** ([10], Definition 3.2) Let \( \Gamma \) be a fractal structure on a metric space \((X, \rho)\), \( F \) be a subset of \( X \), and assume that \( \delta(F, \Gamma_n) \to 0 \). Moreover, let us define the following expression:

\[ \mathcal{H}^s_{n,k}(F) = \inf \left\{ \sum_{i \in I} \operatorname{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_{n,k}(F) \right\}, \]

where

\[ \mathcal{A}_{n,k}(F) = \begin{cases} \{ \{A_i\}_{i \in I} : A_i \in \bigcup_{l \geq n} \Gamma_l, F \subseteq \bigcup_{i \in I} A_i, \text{Card}(I) < \infty \} & \text{if } k = 4 \\ \{ \{A_i\}_{i \in I} : A_i \in \bigcup_{l \geq n} \Gamma_l, F \subseteq \bigcup_{i \in I} A_i \} & \text{if } k = 5, \end{cases} \]

and let also

\[ \mathcal{H}^s_k(F) = \lim_{n \to \infty} \mathcal{H}^s_{n,k}(F), \]

for \( k = 4, 5 \). Then the fractal dimensions IV and V of \( F \) are defined, respectively, by

\[ \dim^k_F(F) = \inf\{s : \mathcal{H}^s_k(F) = 0\} = \sup\{s : \mathcal{H}^s_k(F) = \infty\}, \]

for \( k = 4, 5 \).

In Definition 6.6 as well as in the next one, we shall consider that \( \inf \emptyset = \infty \). For instance, if \( \mathcal{A}_{n,4}(F) = \emptyset \), then \( \dim^4_F(F) = \infty \).
**Definition 6.7:** (10), Definition 3.3  Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, $F$ be a subset of $X$, $\delta > 0$, and assume that $\delta(F, \Gamma_n) \to 0$. Moreover, let us consider the following expression:

$$\mathcal{H}^\varepsilon_{\delta,6}(F) = \inf \left\{ \sum_{i \in I} \text{diam} (A_i)^{\varepsilon} : \{A_i\}_{i \in I} \in \mathcal{A}_{\delta,6}(F) \right\},$$

where

$$\mathcal{A}_{\delta,6}(F) = \left\{ \{A_i\}_{i \in I} : A_i \in \bigcup_{i \in \mathbb{N}} \Gamma_i \text{ for all } i \in I, \text{diam} (A_i) \leq \delta, F \subseteq \bigcup_{i \in I} A_i \right\},$$

and let also

$$\mathcal{H}^\varepsilon_{6}(F) = \lim_{\delta \to 0} \mathcal{H}^\varepsilon_{\delta,6}(F).$$

Then the fractal dimension VI of $F$ is defined by

$$\dim^{\varepsilon}_{\Gamma}(F) = \inf \{s : \mathcal{H}^\varepsilon_{6}(F) = 0\} = \sup\{s : \mathcal{H}^\varepsilon_{6}(F) = \infty\}.$$

Equivalently, from both Definitions 6.6 and 6.7, we have that

$$\mathcal{H}^\varepsilon_{k}(F) = \begin{cases} \infty & \text{if } s < \dim^{\varepsilon}_{\Gamma}(F) \\ 0 & \text{if } s > \dim^{\varepsilon}_{\Gamma}(F), \end{cases}$$

for $k = 4, 5, 6$, provided that $\delta(F, \Gamma_n) \to 0$. The next remark becomes especially useful, since it is not required to consider lower/upper limits (unlike it happens with the box dimension) for $\mathcal{H}^\varepsilon_{k}(F)$ ($k = 4, 5, 6$) calculation purposes, as well as it happens with $\mathcal{H}^\varepsilon_{4}(F)$ and $\mathcal{H}^\varepsilon_{6}(F)$ (see [5, Subsection 2.2]).

**Remark 3:**

1. Since $\mathcal{H}^\varepsilon_{n,k}(F)$ is the general term of a monotonic non-decreasing sequence in $n \in \mathbb{N}$ for $k = 4, 5$, then the fractal dimensions IV and V of any subset $F$ of $X$ always exist.
2. Since $\mathcal{H}^\varepsilon_{5,6}(F)$ is a non-increasing quantity for $s \geq 0$, then we have that $\mathcal{H}^\varepsilon_{6}(F)$ also is by definition, so the fractal dimension VI of any subset $F$ of $X$ always exists.

Let $\dim^{\varepsilon}_{\Gamma}$ denote one of fractal dimensions V or VI, herein. The following theorem contains some properties regarding the behavior of fractal dimensions IV-VI. Their proofs can be found out in [10].

**Theorem 6.8:** Let $\Gamma$ be a fractal structure on a metric space $(X, \rho)$, $F$ be a subset of $X$, and assume that $\delta(F, \Gamma_n) \to 0$.

1. There exist countable subsets $F$ of $X$ such that $\dim^{\varepsilon}_{\Gamma}(F) \neq 0$.
2. $\dim^{\varepsilon}_{\Gamma}(F) = \dim^{\frac{\varepsilon}{2}}_{\Gamma}(F)$.
3. Fractal dimensions V and VI are countably stable, namely, $\dim^{\varepsilon}_{\Gamma}(\bigcup_{i \in I} E_i) = \sup\{\dim^{\varepsilon}_{\Gamma}(E_i) : i \in I\}$, where $\{E_i\}_{i \in I}$ is a countable sequence of subsets of $X$.
4. There exist a subset $F$ of $X$ and a locally finite tiling starbase fractal structure $\Gamma$ with finite order on $X$ such that $\dim^{\varepsilon}_{\Gamma}(F) \neq \dim^{\varepsilon}_{\Gamma}(\overline{F})$.
5. $\dim^{\varepsilon}_{\Gamma}(F) \leq \dim^{\varepsilon}_{\Gamma}(\overline{F}) \leq \dim^{\varepsilon}_{\Gamma}(F)$. 
Such a fixed compact is named the attractor associated with the IFS through an each form fixed under the action of the so-called Hutchinson operator \[ T \]

reader to different directions, unlike similarities. Regarding the theory of self-affine sets, we refer the

reflections, as well. Moreover, a set of a transformations do contract with different ratios, namely,

dimensions in different directions. Unlike similarities. Regarding the theory of self-affine sets, we refer the reader to [6, Section 9.4].

An application to discrete dynamical systems

Self-affine sets constitute a wide range of fractals containing self-similar sets as particular cases. By an affine transformation \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), we shall understand a transformation of the form \( f(x) = T(x) + b \), where \( T \) is a linear transformation on \( \mathbb{R}^n \) that can be represented through an \( n \)-order matrix, and \( b \) is a vector in \( \mathbb{R}^n \). It is worth mentioning that the class of affine transformations contains combinations of translations, rotations, dilations, and reflections, as well. Moreover, affine transformations do contract with differing ratios in different directions, unlike similarities. Regarding the theory of self-affine sets, we refer the reader to [6, Section 9.4].

Let \( \mathcal{F} = \{f_1, \ldots, m\} \) be an iterated function scheme (IFS, for short), where all the mappings \( f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d \) are contractions, namely, they satisfy that \( d(f_i(x), f_i(y)) \leq c_i \cdot d(x, y) \) for all \( x, y \in \mathbb{R}^d \) and all \( i = 1, \ldots, m \), and \( c_i \) is the contraction ratio associated with each \( f_i \). It is well-known that there exists a unique non-empty compact set \( \mathcal{K} \) that remains fixed under the action of the so-called Hutchinson operator \[ H \]

\[ \mathcal{K} = \bigcup_{i=1}^{m} f_i(\mathcal{K}). \] Such a fixed compact is named the attractor associated with the IFS \( \mathcal{F} \). Further, it also holds that \( \mathcal{K} = \bigcap_{i=0}^{\infty} F^i(E) \), where \( F(A) = \bigcup_{i=1}^{m} f_i(A) \) denotes the Hutchinson operator, and

\[ \dim_H(F) \leq \dim_H^6(F) \leq \dim_H^5(F) \leq \dim_H^3(F) \leq \dim_H^2(F) \leq \dim_H^1(F). \]

If \( \Gamma \) is finite, then \( \dim_H(F) \leq \dim_H^6(F) \leq \dim_H^5(F) \leq \dim_H^4(F) \leq \dim_H^3(F) \leq \dim_H^2(F) \leq \dim_H^1(F) \).

If \( \Gamma \) is diameter-positive (see [10, Definition 3.6]), then \( \dim_H^5(F) = \dim_H^1(F) \).

If \( \Gamma \) is the natural fractal structure on \( X = \mathbb{R}^d \), then

(a) \( \dim_H(F) = \dim_H^6(F) = \dim_H^5(F) \leq \dim_H^3(F) = \dim_H^2(F) = \dim_H^1(F) = \dim_B(F) \).

(b) If \( F \) is compact, then \( \dim_H(F) = \dim_H^6(F) = \dim_H^5(F) = \dim_H^4(F) \leq \dim_H^3(F) = \dim_H^2(F) = \dim_H^1(F) = \dim_B(F) \).

(c) If \( F \) is bounded, then \( \dim_H(F) = \dim_H^4(F) \).

Figure 1. Graphical approach to the self-affine set \( \mathcal{K} \) defined along Section 7.
A refers to any non-empty compact set such that \( f_i(A) \subset A \), for all \( i = 1, \ldots, m \) (see [6, Theorem 9.1]). In particular, if all the \( f_i \) are affine contractions then \( K \) is called a self-affine set.

So let \( K \subset \mathbb{R}^2 \) be the self-affine set uniquely determined by the IFS \( \mathcal{F} = \{f_1, f_2, f_3\} \), where each affine contraction is defined next:

1. \( f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), such that \( f_1(x, y) = (\omega_1, \omega_2) \), and
   \[
   \begin{pmatrix}
   \omega_1 \\
   \omega_2
   \end{pmatrix} =
   \begin{pmatrix}
   \cos \theta - \sin \theta \\
   \sin \theta & \cos \theta
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix} +
   \begin{pmatrix}
   a_1 \\
   b_1
   \end{pmatrix},
   \]
   where \( k \in (0, 1) \).
2. \( f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), such that \( f_2(x, y) = (\omega_1, \omega_2) \), and
   \[
   \begin{pmatrix}
   \omega_1 \\
   \omega_2
   \end{pmatrix} =
   \begin{pmatrix}
   \kappa_1 & 0 \\
   0 & \kappa_2
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix},
   \]
   where \( \kappa_1, \kappa_2 \in (0, 1) \).
3. \( f_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), such that \( f_3(x, y) = (\omega_1, \omega_2) \), and
   \[
   \begin{pmatrix}
   \omega_1 \\
   \omega_2
   \end{pmatrix} =
   \begin{pmatrix}
   \kappa_3 & -\kappa_3 \\
   0 & \kappa_3
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix} +
   \begin{pmatrix}
   a_2 \\
   b_2
   \end{pmatrix},
   \]
   where \( \kappa_3 \in (0, 1) \).

Notice that \( \mathcal{F} \) gives rise to a whole family of self-affine sets which depends on several parameters. In this case, the parameter choice has been as follows: \( \kappa_1 = 1/2, \kappa_2 = 1/4, \kappa_3 = 1/2, a_1 = 1/2, a_2 = 3/2, b_1 = 3/4, b_2 = 0 \), and \( \theta = \pi/4 \). For illustration purposes, Figure 1 provides a graphical approach to the self-affine set \( K \). Let \( F \) denote the Hutchinson operator associated with the IFS \( \mathcal{F} \), namely, \( F(A) = \bigcup_{f \in \mathcal{F}} f(A) \), for all compact subset \( A \subset \mathbb{R}^2 \). We shall consider the discrete dynamical system defined by

\[
p_n = F(p_{n-1}),
\]

where \( p_0 = (x_0, y_0) \in \mathbb{R}^2 \) refers to the initial point in such a construction. Notice that \( K \) is not a self-similar set, namely, all the \( f_i \) in \( \mathcal{F} \) are not similarities but affine contractions, instead. Thus, the classical Moran’s Theorem (recall [6, Theorem 9.3]) cannot be applied for box dimension calculation purposes. So let us algorithmically calculate the box dimension of the self-affine set \( K \) in order to explore the chaotic behavior of that dynamical system. In this way, we have obtained that \( \dim_B(K) = 1.33995 \), where the calculations have been carried out via a fractal dimension 1 algorithm (see Fractal dimension approach 3.3), and under the assumption that \( \Gamma \) is the natural fractal structure on \( \mathbb{R}^2 \). Further, Theorem 5.3 (5.3) allows to affirm that \( \dim_B(K) = \dim_1(K) = \dim_2(K) = \dim_3(K) = 1.33995 \), and that result provides an upper bound for the Hausdorff dimension of \( K \) (and hence, to \( \dim_0(K) = \dim_5(K) = \dim_4(K) \), since \( K \) is compact). It is worth noting that such a fractal dimension value remains quite stable for different sizes of \( K \) (namely, the number of points that have been considered in order to approach the actual self-affine set \( K \)).
Regarding the complexity of the self-affine set \( \mathcal{K} \) (and hence, the complexity of that discrete dynamical system), it holds that \( N_n(\mathcal{K}) \cong 2^n \dim_B(\mathcal{K}) \), or equivalently, \( N_{n+1}(\mathcal{K}) \cong 2^{\dim_B(\mathcal{K})} \cdot N_n(\mathcal{K}) \) for all \( n \in \mathbb{N} \). Roughly speaking, this means that for each element in level \( n \) of \( \Gamma \) that meet \( \mathcal{K} \), there are about \( 2^{\dim_B(\mathcal{K})} \) elements in the next level of \( \Gamma \).

**Funding**

This work was supported by the Fundación Séneca – Agencia de Ciencia y Tecnología de la Región de Murcia [19219/PI/14]; Spanish Ministry of Economy and Competitiveness [MTM2014-51891-P].

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**References**
